Numerical Solution and Stability for Model of Extensible Beam

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ABSTRACT- In this paper, numerical methods (finite differences methods for explicit and implicit) has been applied to solve nonlinear partial differential equations. In methodology, the beam was divided into very smaller squares, then the study discussed three partial differential equations generating from model. The first equation called longitudinal vibrations of a beam, second equation known as transverse vibrations of a beam and then the third equation considered the extensible beam. The equation of extensible beam was defined by Woiniwsky- Krieger as a model for transverse deflection of an extensible beam of natural length. The study discussed the stability of these models (longitudinal vibrations, transverse vibrations and extensible beams). The stability solution has been counted and considered unconditionally for implicit method, but it’s conditional for an explicit method. Obtaining the stability and convergent solution for longitudinal vibrations of a beam if width divisions is less than length divisions (R < 2), and for transverse vibrations of a beam if width divisions less than the square length divisions (R < 0.25), as well as for extensible beam if width divisions less than the square length divisions, the study recommended to use an implicit method. But in case of using an explicit method, the divisions must be adhered for a stable and convergent solution.

Keywords: Partial Differential Equations, Finite Differences, Beam, MATLAB Programming.

INTRODUCTION
Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering [1]. A beam is a bar-like structural member whose primary function is to support transverse loading and carry it to the supports, this equation describes the motion of a beam initially located on the x-axis which is vibrating transversely "perpendicular to the x-direction", in this case u(x, t) is the transverse displacement or deflection at any time t of any points x [2].

In the recent literature the behavior of a clamped free non-linear inextensible Euler Elastic introduced in Euler [9], see Luongo and Zulli [10], Eugster [11], Steigmann and Faulkner [12] for general reference works, has been mathematically investigated under distributed load (2016) [3]. In particular, the set of stable equilibrium configurations has been completely characterized in Della Corte et al (2019) [4]. Today we get the numerical solution is very important especially for nonlinear models, because the traditional methods for solving nonlinear
The local truncation error for this equation is

\[ t_{ij} = \frac{\kappa^2}{12} \frac{\partial^4 u(x_i, \eta_j)}{\partial \eta^4} - \frac{h^2}{12} \frac{\partial^4 u(\xi_i, t_j)}{\partial \xi^4} = O(k^2) + O(h^2) \]  

\[ \beta = 0 \& \gamma = 0, \text{ equation (1), is called transverse vibrations of a beam.} \]

Consider the initial- boundary value problem at both ends. Consider the boundary conditions:

\[ u(0, t) = A, \quad u(L, t) = B, \]

\[ u_x(0, t) = C \& u_x(L, t) = D \]

Initial conditions:

\[ u(x, 0) = f(x) \quad \& \quad u_t(x, 0) = g(x) \]

### TABLE 1: Important Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mining</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>Is ratio between young’s modulus multiply by cross sectional second moment area and density</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Is ratio between young’s modulus multiply by cross sectional area and density multiply by length</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Is ratio between young’s modulus multiply by cross sectional area and density multiply by length</td>
</tr>
</tbody>
</table>

**MATERIALS AND METHODS**

In this paper, we use finite differences methods (Explicit Method & Implicit Method), the finite differences approximations for derivatives are one of the simplest and of the oldest methods to solve differential equations. L. Euler knew it, in this paper, we using the explicit method to approximate the derivatives for central operator difference [8].

**STABILITY ANALYSIS**

The stability analysis is giving optimal option to choose the parameters to obtain best approximate solution.

**Stability of explicit method for longitudinal vibrations of a beam equation.**

\[ \frac{\partial^2 u}{\partial t^2} - \beta \frac{\partial^2 u}{\partial x^2} = 0 \]  

\[ (\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}) - \beta (\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}) = O(k^2) + O(h^2) \]

The local truncation error for this equation is

\[ t_{ij} = \frac{h^2}{12} \frac{\partial^4 u(x_i, \eta_j)}{\partial \eta^4} - \frac{\kappa^2}{12} \frac{\partial^4 u(\xi_i, t_j)}{\partial \xi^4} = O(k^2) + O(h^2) \]  

\[ u_{i,j+1} = ru_{i+1,j} + (2 - 2r)u_{i,j} + ru_{i-1,j} - u_{i,j-1} \]
where  \( r = \frac{\beta k^2}{h^4} \), R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by \( \beta \).

Let us:  \( u_{i,j} = (-1)^{i} \lambda^{j} \) or  \( u_{i,j} = \lambda^{j} e^{ix\theta} \)  

\[
(-1)^{i} \lambda^{j+1} = r(-1)^{i+1} \lambda^{j} + (2 - 2r)(-1)^{i} \lambda^{j} + r(-1)^{j-1} \lambda^{i} - (-1)^{i} \lambda^{j-1}  
\]

Multiply both sides by, \((-1)^{i} \lambda^{-j}\), we obtain
\[
\lambda = -r + (2 - 2r) - r - \lambda^{-1}  
\]  
(11)  
\[
\lambda + \lambda^{-1} = 2 - 2r  
\]  
(12)
\[
\Rightarrow \frac{\lambda^2 + 1}{\lambda} = 2 - 2r  
\]  
(13)
Suppose  \( w = 1 - r \) \( \Rightarrow \)  \( \lambda^2 - 2w\lambda + 1 = 0 \)  
\[
\Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - 1}  
\]  
(14)
Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation, we may conclude that  \( \lambda_1 \lambda_2 = 1 \). However, for stability of solutions we require  \(|\lambda_1| \leq 1 \) and  \(|\lambda_2| \leq 1 \). Given the constraint  \( \lambda_1 \lambda_2 = 1 \), the only possibility if the solution to be stable is  \(|\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies \([6]\).

\[
|w| = |1 - r| < 1 \Rightarrow |r - 1| < 1 , \Rightarrow \ r < 2 , \ r = \frac{\beta k^2}{h^2} < 2 \Rightarrow k^2 < \frac{2h^2}{\beta}  
\]
The stability of explicit method is conditionally.

**Stability of implicit method for longitudinal vibrations of a beam equation.**

\[
\frac{\partial^2 u}{\partial t^2} - \beta \frac{\partial^2 u}{\partial x^2} = 0  
\]  
(15)
\[
\left( \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} \right) - \beta \left( \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^4} \right) = O(k^2) + O(h^2)  
\]  
(16)
The local truncation error for this equation is
\[
t_{ij} = \frac{k^2}{12} \frac{\partial^4 u}{\partial t^4}(\xi_i, \eta_j) - \frac{\beta h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, \eta_j) = O(k^2) + O(h^2)  
\]  
(17)
\[
2u_{i,j} = -ru_{i+1,j+1} + (1 + 2r)u_{i,j+1} - ru_{i-1,j+1} + u_{i,j-1}  
\]  
(18)
where  \( r = \frac{\beta k^2}{h^4} \). R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by \( \beta \).

Let us:  \( u_{i,j} = (-1)^{i} \lambda^{j} \) or  \( u_{i,j} = \lambda^{j} e^{ix\theta} \)  

\[
2(-1)^{i} \lambda^{j} = -r(-1)^{i+1} \lambda^{j+1} + (1 + 2r)(-1)^{i} \lambda^{j+1} - r(-1)^{j-1} \lambda^{i+1} + (-1)^{j} \lambda^{j-1}  
\]
Multiply both sides by, \((-1)^{-1} \lambda^{-j}\), we obtain
\[
2 = r\lambda + (1 + 2r)\lambda + r\lambda + \lambda^{-1}  
\]  
(20)
\[
(1 + 4r)\lambda + \lambda^{-1} = 2  
\]  
(21)
\[
\Rightarrow \frac{(1 + 4r)\lambda^2 + 1}{\lambda} = 2  
\]  
(22)
Suppose  \( w = \frac{1}{1+4r} \) \( \Rightarrow \)  \( \lambda^2 - 2w\lambda + w = 0 \) \( \Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - w} \)  
(23)
Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation. However, for stability of solutions we require  \(|\lambda_1| \leq 1 \) and  \(|\lambda_2| \leq 1 \). The only possibility, if the solution to be stable is  \(|\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies \([6]\).

\[
|w| = \left| \frac{1}{1+4r} \right| < 1 \Rightarrow |1 + 4r| > 1 \ , \Rightarrow \ r > 0 , \ r = \frac{\beta k^2}{h^2} > 0 \Rightarrow k > 0  
\]
The stability of implicit method is unconditionally.

**Stability of explicit method for transvers vibrations of a beam equation.**

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} = 0  
\]  
(24)
\[
\left( \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} \right) + c^2 \left( \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} \right) = O(k^2) + O(h^2)  
\]  
(25)
The local truncation error for this equation is
\[
t_{ij} = \frac{k^2}{12} \frac{\partial^4 u(x_i, \eta_j)}{\partial t^4} + \alpha \frac{2h^2}{8!} \frac{\partial^8 u(x_i, t_j)}{\partial x^8} = O(k^2) + O(h^2),
\]
(26)

\[
u_{i,j+1} = -ru_{i+2,j} + 4ru_{i+1,j} + 2(1 - 3r)u_{i,j} + 4ru_{i-1,j} - ru_{i-2,j} - u_{i,j-1}
\]
(27)

where \( r = \frac{ak^2}{h^4} \). \( R \) is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by \( \alpha \). Let us: \( u_{i,j} = (-1)^i \lambda^j \) or \( u_{i,j} = \lambda^j e^{i\pi x} \)

\[
(-1)^j \lambda^{j+1} = -r(-1)^j + 4r(-1)^j + 2(1 - 3r)(-1)^j \lambda^j + 4r(-1)^j \lambda^j - r(-1)^j \lambda^j - (-1)^j \lambda^{j-1}
\]

Multiply both sides by, \((−1)^{−i} λ^{−j}\), we obtain

\[
\lambda = -r - 4r + 2(1 - 3r) - 4r - r - \lambda^{-1}
\]

\[
\lambda + \lambda^{-1} = 2 - 16r
\]

\[
\frac{\lambda}{\lambda + 1} = 2 - 16r
\]

(28)

(29)

(30)

(31)

Suppose \( w = 1 - 8r \) \( \Rightarrow \lambda^2 - 2w\lambda + 1 = 0 \). \( \Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - 1} \)

(32)

Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation, we may conclude that \( |\lambda_1| \leq 1 \) and \( |\lambda_2| \leq 1 \). Given the constraint \( \lambda_1,\lambda_2 = 1 \), the only possibility if the solution to be stable is \( |\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies [6].

\[
|w| = |1 - 8r| < 1 \Leftrightarrow |8r - 1| < 1, \quad \Rightarrow \quad r < \frac{1}{4}, \quad r = \frac{ak^2}{h^4} < \frac{1}{4} \quad \Rightarrow \quad k^2 < \frac{h^4}{4a^2}
\]

The stability of explicit method is conditionally.

**Stability of implicit method for transvers vibrations of a beam equation.**

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} = 0
\]

(33)

\[
\left( \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \right) + c^2 \left( \frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4} \right) = O(k^2) + O(h^2)
\]

(34)

The local truncation error for this equation is

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} = 0
\]

\[
\left( \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \right) + c^2 \left( \frac{u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1}}{h^4} \right) = O(k^2) + O(h^2)
\]

(35)

Let us: \( u_{i,j} = (-1)^i \lambda^j \) or \( u_{i,j} = \lambda^j e^{i\pi x} \)

\[
2(-1)^j \lambda^j = r(-1)^j + 4r(-1)^j + 2(1 + 6r)(-1)^j \lambda^j + 4r(-1)^j \lambda^j - r(-1)^j \lambda^j - (-1)^j \lambda^{j-1}
\]

Multiply both sides by, \((−1)^{−i} λ^{−j}\), we obtain

\[
2 = r\lambda + 4r + (1 + 6r)\lambda + 4r\lambda + r\lambda + \lambda^{-1}
\]

\[
(1 + 16r)\lambda + \lambda^{-1} = 2
\]

\[
(1 + 16r)\lambda + \lambda^{-1} = 2
\]

(37)

(38)

(39)

(40)

Suppose \( w = \frac{1}{1 + 16r} \) \( \Rightarrow \lambda^2 - 2w\lambda + w = 0 \). \( \Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - w} \)

(41)

Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation. However, for stability of solutions we require \( |\lambda_1| \leq 1 \) and \( |\lambda_2| \leq 1 \). The only possibility, if the solution to be stable is \( |\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies [6].

\[
|w| = \left| \frac{1}{1 + 16r} \right| < 1 \Rightarrow |1 + 16r| > 1, \quad \Rightarrow \quad r > 0, \quad r = \frac{ak^2}{h^4} > 0 \quad \Rightarrow \quad k > 0
\]

The stability of explicit method is unconditionally.

**Stability of explicit method for extensible beam equation.**

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial u^2}{\partial x} \frac{\partial^2 u}{\partial x^2} = 0
\]

(42)
\[
\begin{align*}
(r_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + \alpha \left( u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j} \right) \\
\quad - \beta \left( u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right) - \gamma \left( u_{i+1,j} - u_{i-1,j} \right)^2 \left( u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right) \\
= 0 \\
\end{align*}
\]

\[
\begin{align*}
u_{i,j+1} = -ru_{i+2,j} + 4ru_{i+1,j} + (2 - 6r)u_{i,j} + 4ru_{i-1,j} - ru_{i-2,j} \\
+ \frac{\beta h^2}{r} \left( ru_{i+1,j} - 2ru_{i,j} + ru_{i-1,j} \right) \\
+ \frac{\gamma}{4\alpha} \left( ru_{i+1,j} - ru_{i-1,j} \right)^2 \left( ru_{i+1,j} - 2ru_{i,j} + ru_{i-1,j} \right) - u_{i-1,j} 
\end{align*}
\]

where \( r = \frac{ak^2}{h^2} \). R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by \( \alpha \).

Let us: \( u_{i,j} = (-1)^i \lambda^j \) or \( u_{i,j} = \lambda^j e^{j\Delta x \theta} \)

\[
\begin{align*}
(-1)^i \lambda^{j+1} &= -r(-1)^{i+2} \lambda^j + 4r(-1)^{i+1} \lambda^j + (2 - 6r)(-1)^i \lambda^j + 4r(-1)^{i-1} \lambda^j - r(-1)^{i-2} \lambda^j \\
+ \frac{\beta h^2}{r} \left( r(-1)^{i+1} \lambda^j - 2r(-1)^i \lambda^j + r(-1)^{i-1} \lambda^j \right) \\
+ \frac{\gamma}{4\alpha} \left( r(-1)^{i+1} \lambda^j - r(-1)^{i-1} \lambda^j \right)^2 \left( r(-1)^i \lambda^j + r(-1)^{i-1} \lambda^j \right) \\
- (-1)^i \lambda^{j-1} 
\end{align*}
\]

Multiply both sides by, \((-1)^{-i} \lambda^{-j}\), we obtain

\[
\begin{align*}
\lambda &= -r - 4r + (2 - 6r) - 4r - r - \frac{4\beta h^2}{\alpha} r - \lambda^{-1} \\
\lambda + \lambda^{-1} &= 2 - 16r - \frac{4\beta h^2}{\alpha} r \\
\lambda^2 + 1 &= 2 - 16r - \frac{4\beta h^2}{\alpha} r \\
\lambda &= 1 - 8r - \frac{2\beta h^2}{\alpha} r \\
\lambda^2 - 2\lambda + 1 &= 0, \Rightarrow \lambda_{1,2} &= w \pm \sqrt{w^2 - 1} 
\end{align*}
\]

Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation, we may conclude that \( \lambda_1 \lambda_2 = 1 \). However, for stability of solutions we require \( |\lambda_1| \leq 1 \) and \( |\lambda_2| \leq 1 \). Given the constraint \( \lambda_1 \lambda_2 = 1 \), the only possibility if the solution to be stable is \( |\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies \[6\].

\[
|w| = \left| 1 - 8r - \frac{2\beta h^2}{\alpha} r \right| < 1 \Rightarrow \left| (8 + \frac{2\beta h^2}{\alpha}) r - 1 \right| < 1 \Rightarrow r < \frac{1}{(4\beta h^2/\alpha)}
\]

\[
\begin{align*}
\frac{ak^2}{h^2} < \frac{1}{(4\beta h^2/\alpha)} \Rightarrow k^2 < \frac{h^4}{(4\alpha + \beta h^2)}
\end{align*}
\]

The stability of explicit method is conditionally.

**Stability of implicit method for extensible beam equation.**

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{\partial x^4} - \frac{\beta}{\partial x^2} - \gamma \left( \frac{\partial u}{\partial x} \right)^2 \frac{\partial^2 u}{\partial x^2} = 0 \\
\end{align*}
\]

\[
\begin{align*}
\left( u_{i,j+1} - 2u_{i,j} + u_{i,j-1} \right) + \alpha \left( u_{i+2,j+1} - 4u_{i+1,j+1} + 6u_{i,j+1} - 4u_{i-1,j+1} + u_{i-2,j+1} \right) \\
- \beta \left( u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \right) \\
- \gamma \left( \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} \right)^2 \left( u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \right) = 0 \\
\end{align*}
\]

\[
\begin{align*}
- \beta h^2 \left( ru_{i+1,j+1} - 4ru_{i,j+1} + (1 + 6r)u_{i,j+1} - 4ru_{i-1,j+1} + ru_{i-2,j+1} \right) \\
- \gamma \left( ru_{i+1,j+1} - ru_{i-1,j+1} \right)^2 \left( ru_{i+1,j+1} - 2ru_{i,j+1} + ru_{i-1,j+1} \right) + u_{i,j-1}
\end{align*}
\]
where \( r = \frac{a k^2}{h^4} \). R is ratio between square of step length of time (k) and step length of a beam to power four (h) multiply by \( \alpha \). Let us: \( u_{i,j} = (-1)^i \lambda^j \) or \( u_{i,j} = \lambda^i e^{\alpha x \theta} \)

\[
2(-1)^i \lambda^j = r(-1)^i + 2 \lambda^j + 1 - 4 r(-1)^i \lambda^j + 1 + (1 + 6 r)(-1)^i \lambda^j + 1 - 4 r(-1)^i - 1 \lambda^j + 1
+n r(-1)^i - 2 \lambda^j + 1 - 2 r(-1)^i \lambda^j + 1 + r(-1)^i - 1 \lambda^j + 1
\]

\[
-\frac{\gamma}{4 r} (r(-1)^i 2 \lambda^j + 1 - 2 r(-1)^i \lambda^j + 1)^2 (r(-1)^i \lambda^j + 1 - 2 r(-1)^i \lambda^j + 1 + r(-1)^i - 1 \lambda^j + 1) + r(-1)^i - 1 \lambda^j + 1 + (-1)^i \lambda^j - 1
\]

Multiply both sides by, \((-1)^i \lambda^{-j}\), we obtain

\[
2 = r \lambda + 4 r \lambda + (1 + 6 r) \lambda + r \lambda + 4 \frac{\theta h^2}{\alpha} r \lambda + \lambda^{-1}
\]

\[
\left(1 + 16 r + \frac{4 \theta h^2}{\alpha} \right) \lambda + \lambda^{-1} = 2
\]

\[
(1 + 16 r + 4 \theta r^2) \lambda^2 + 1 = 2 \lambda
\]

Suppose \( w = \frac{1}{1 + 16 r + 4 \theta r^2} \Rightarrow \lambda^2 - 2 \lambda w + w = 0, \Rightarrow \lambda_{1,2} = w \pm \sqrt{w^2 - w} \)

Now since \( \lambda_1 \) and \( \lambda_2 \) are roots of this quadratic equation. However, for stability of solutions we require \( |\lambda_1| \leq 1 \) and \( |\lambda_2| \leq 1 \). The only possibility, if the solution to be stable is \( |\lambda_1| = |\lambda_2| = 1 \), thus \( \lambda \) must fall on the unit disk, which implies \([6]\).

\[
|w| = \left| \frac{1}{1 + 16 r + \frac{4 \theta h^2}{\alpha} r} \right| < 1 \Rightarrow \left| 1 + \left( 16 + \frac{4 \theta h^2}{\alpha} \right) r \right| > 1 \Rightarrow r > 0, r = \frac{a k^2}{h^4} > 0
\]

The stability of implicit method is unconditionally.

**Algorithm and Numerical Results**

**Algorithm of equation (1) for applied explicit method**

To obtain the numerical solution of equation 1.

**Input:** endpoint L; maximum time T; constants \( \alpha, \beta, \gamma \); integers \( n \) and \( m \)

**Output:** approximations \( u(x_i, t_j) \), for each \( i = 0,1,\ldots,m \) and \( j = 0,1,\ldots,n \)

**Step 1:** \( h = \frac{L}{n} \),\n
\( k = \frac{T}{m} \)

\( r = \alpha \cdot k^2 h^4 \)

\( p = \gamma \cdot k^2 (2 h)^2 \)

**Step 2:** for \( i = 0,1,\ldots,m \)

for \( j = 0,1,\ldots,n \)

Do step 3 and step 4

**Step 3:** \( u(x_0, t_j) = A \)

\( u(x_n, t_j) = B \)

**Step 4:** \( u(x_i, t_0) = f(x_i) \)

**Step 5:** for \( i = 1,\ldots,n-1 \)

for \( j = 1,\ldots,m-1 \)

\( u(x_{i-1}, t_j) = -ru(x_{i-1}, t_j) + 4ru(x_{i+1}, t_j) + 2(1 - 3 r)u(x_i, t_j) + 4ru(x_{i-1}, t_j) - ru(x_{i+2}, t_j) - u(x_i, t_{j-1}) + \left( \beta k^2 + p \left( u(x_{i+1}, t_j) - u(x_{i-1}, t_j) \right) \right)^2 (u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j))/h^2 \)

**Step 6:** output \( u_{00}, u_{01}, u_{02}, \ldots, u_{nn} \)

**Step 7:** Stop (the producer is complete)

**Algorithm of equation (1) for applied implicit method**
To obtain the numerical solution of equation 1.

**Input:** endpoint L; maximum time T; constants $\alpha, \beta, \gamma$; integers $n$ and $m$

**Output:** approximations $u(x_i, t_j)$, for each $i=0,1,\ldots,m$ and $j=0,1,\ldots,n$

**Step 1:** $h=L/n$

$k=T/m$

$r=\alpha \cdot k^2/h^2$

$p=\gamma \cdot k^2/(2h)^2$

**Step 2:** for $i=0,1,\ldots,m$

for $j=0,1,\ldots,n$

Do step 3 and step 4

**Step 3:**

$u(x_0, t_j) = A$

$u(x_n, t_j) = B$

**Step 4:**

$u(x_i, t_0) = f(x_i)$

**Step 5:** for $i=1,\ldots,m-1$

for $j=1,\ldots,n-1$

$$ru(x_{i+1}, t_{j+1}) - 4ru(x_{i+1}, t_{j+1}) + (1 + 6r)u(x_i, t_{j+1}) - 4ru(x_{i-1}, t_{j+1}) + ru(x_{i-2}, t_{j+1}) + \left(\frac{\beta k^2 + p}{h^2}(u(x_{i+1}, t_{j+1}) - u(x_i, t_{j+1}))^2\right) * \frac{u(x_{i+1}, t_{j+1}) - 2u(x_i, t_{j+1}) + u(x_{i-1}, t_{j+1})}{h^2} = 2u(x_i, t_{j+1}) + u(x_i, t_{j-1})$$

**Step 6:** output $u_{00}, u_{01}, u_{02}, \ldots, u_{nn}$

**Step 7:** Stop (the producer is complete)

**Example 1:** Consider length of x-axis equal 10 and width equal 5. $n=6, m=6, (x) = \sin(x), g(x) = x, A = 1, B = 3, C = 0, D = 0 & \beta = 0.5$. In longitudinal vibrations of a beam.

**Table 2:** Approximate Solution By Using Explicit Method

<table>
<thead>
<tr>
<th>x</th>
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**Table 3:** Approximate Solution By Using Implicit Method

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**Table 4:** Error Estimation For Longitudinal Vibrations Of A Beam

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Error=$|u^{k+1} - u^k|$
Figure 1: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method is unstable when $R \geq 2$ and is stable at $R < 2$

Figure 2: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method

### Table 5: Approximate Solution by Using Explicit Method

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### Table 6: Approximate Solution by Using Implicit Method

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### Table 7: Error Estimate for Transverse of Vibrations of A Beam

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<tr>
<th>Explicit Method</th>
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</table>

Error=$|u^{k+1} - u^k|$
Figure 3: Graphical Representation of transverse vibrations of a beam equation is unstable when $R \geq 0.25$ and is stable at $R < 0.25$.

Figure 4: Graphical Representation of longitudinal vibrations of a beam equation by using implicit method.

Figure 5: Graphical Representation of extensible beam equation is unstable when $R = 0.1800$ and is stable at $R = 0.0009$.

Table 8: Approximate Solution by Using Explicit Method

<table>
<thead>
<tr>
<th>x</th>
<th>t=0.000</th>
<th>t=0.8333</th>
<th>t=1.6667</th>
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</table>

Table 4, shown the error is smaller in stability case. Figure 1, shown the solution is convergent in stability case, but is divergent in instability case, from figure 2 we get a solution is convergent at all case.

Example 2: Consider length of x-axis equal 10 and width equal 5. n=6, m=6, $f(x) = \sin(x)$, $g(x) = x$, $A = 1, B = 3, C = 0, D = 0 & \alpha = 0.5$. In Transverse of vibrations of a beam.
Table 7, shown the error is smaller in stability case. Figure 3, shown the solution is convergent in stability case, but is divergent in instability case, from figure 4 we get a solution is convergent at all case.

**Example 3:** Consider length of x-axis equal 10 and width equal 5. \( n=6, m=6, f(x) = \sin(x), g(x) = x, A = 1, B = 3, C = 0, D = 0, \alpha = 0.01, \beta = 0.02 \) & \( \gamma = 0.03 \). In extensible beam.

Table 9, shown the error is smaller in stability case. Figure 5, shown the solution is convergent in stability case, but in instability case, the solution is divergent.

**Table 9: Error Estimation for Extensible Beam**

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Error=\( ||u^{k+1} - u^k|| \)

**CONCLUSIONS**

In this paper, the study discussed solutions of extensible beam linear or and nonlinear partial differential equation dependent for parameter, by using finite difference methods also we discuss the stability we get i) The stability of implicit method unconditionally but the stability of explicit method is conditionally. ii) The explicit method of longitudinal vibrations of a beam equation is stable if \( R < 2 \) and unstable when \( R \geq 2 \). iii) The explicit method of Transverse vibrations of a beam equation is stable if \( R < 0.25 \) and unstable when \( R \geq 0.25 \), iv) The explicit method of extensible beam equation is stable if \( R < \frac{\alpha}{(4\alpha+\beta h^2)} \) and unstable when \( R \geq \frac{\alpha}{(4\alpha+\beta h^2)} \). v) The implicit method of longitudinal vibrations of a beam equation, Transverse vibrations of a beam equation and extensible beam equation are stable for any value of \( R \). vi) From tables 4, 7 & 9 we get the error is very small when we applied implicit method, but in explicit method we get small error use stability case, vii) From figures 1, 2 & 3, at stability case for explicit method and implicit method the figures is similar and uniform, but in instability case the figures are not similar and differences, viii) Future work, we hop the research applied the implicit method for solving, but sometime the implicit method for nonlinear model is very difficult to compute solution in this case applied the explicit method and choose the parameters to give stability.

**REFERENCES**